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It has been recently shown that a certain non-topological spin foam model can be obtained from the Feynman expansion of a field theory over a group. The field theory defines a natural “sum over triangulations”, which removes the cut off on the number of degrees of freedom and restores full covariance. The resulting formulation is completely background independent: spacetime emerges as a Feynman diagram, as it did in the old two-dimensional matrix models. We show here that *any* spin foam model can be obtained from a field theory in this manner. We give the explicit form of the field theory action for an arbitrary spin foam model. In this way, any model can be naturally extended to a sum over triangulations. More precisely, it is extended to a sum over 2-complexes.

I. INTRODUCTION

The spin foam formalism [1–7] is a beautiful convergence of different approaches to general relativistic –or background independent– quantum field theory, and in particular to quantum gravity. A surprising variety of theories admit a spin foam formulation. Among these are: topological field theories [8–10]; modifications of topological quantum field theories related to quantum general relativity [11–13]; loop quantum gravity [14,15], where spin foams appear as histories of spin networks [4,5]; lattice formulations of covariant theories [13,16]; and causal spin-networks [17]. Spin foams seem therefore to represent a rather general tool for dealing with background independent quantum field theories, or general relativistic quantum field theories [18], and thus for describing quantum spacetime [19].

The aspect of the spin foam formalism which is less understood is the “sum over triangulations” (or the “fine triangulation limit”) needed to restore full general covariance when the model is not topological – that is to say, in the physically interesting case. In a recent work [20], this problem is addressed in the context of a specific model, the Barrett-Crane (BC) model [11]. It is shown in [20] that the BC model can be obtained from the perturbative expansion of a field theory over a group. The BC model partition function over a given triangulation Δ is precisely the Feynman amplitude of a certain Feynman graph determined by Δ . The full Feynman expansion of the field theory determines then a natural generalization of the model to a “sum over triangulations”, restoring the infinite number of degrees of freedom and covariance. In the present paper, we show that the same reformulation can be obtained for *any* spin foam model. Given an arbitrary spin foam model, we give here an explicit algorithm for constructing a field theory whose Feynman expansion gives back the spin foam model. This procedure defines an extension to a “sum over triangulation” for any spin foam model.

More precisely, we find that the relevant objects are not triangulations (hence the quotation marks, above), but weaker structures: 2-complexes. This fact confirms previous indications that 2-complexes are the correct objects on which to formulate spin foam models. A triangulation determines a 2-complex: the 2-skeleton of its dual cellular complex; but the converse is not true. Spin foam models considered so far have been generally defined over triangulations; however, they actually depend on the 2-complex only and not the full triangulation. The same is true for lattice discretizations of generally covariant theory. Furthermore, the spin foam expansion of canonical loop quantum gravity is in terms of 2-complexes, not triangulations [4,5]. 2-complexes, rather than triangulations, appear thus to be the natural tool in general relativistic quantum field theory.

The paper is organized as follows. In Section II, we give the general definition of the class of models we consider. Section III contains our main result: we show that all models in this class can be obtained from the Feynman expansion of a field theory, and we give the explicit algorithm for constructing the action of the field theory from the vertex amplitude of the spin foam model. In Section IV, we connect our formalism to lattice gauge theory. In particular, following [13], we show how each spin foam model can be viewed as a generally covariant version of a lattice gauge theory. From this perspective, the sum over 2-complexes determined by the field theory is a way of implementing the limit in which the cutoff induced by the triangulation is removed. In section V we indicate how to extend our result to more complex models, and we present some conclusive remarks.

II. SPIN FOAM MODELS

We consider models defined by the formal sum

$$Z = \sum_J N(J) \sum_c \prod_{f \in J} \dim a_f \prod_{v \in J} A_v(c). \quad (1)$$

J is a 2-complex. A two-complex is a (combinatorial) set of elements called “vertices” v , “edges” e and “faces” f , and a boundary relation among these, such that an edge is bounded by two vertices, and a face is bounded by a cyclic sequence of contiguous edges (edges sharing a vertex).

Given a triangulation Δ of a four dimensional manifold, a 2-complex $J(\Delta)$ is defined as the 2-skeleton of the cellular complex dual to Δ . That is, we can identify the vertices v with the 4-simplices of Δ , the edges e with the tetrahedra and the faces f with the triangles of Δ . Notice that each vertex of $J(\Delta)$ bounds precisely five edges and ten faces, and each edge bounds precisely four faces. We say that vertices of this form are 5-valent, and that edges of this form are 4-valent.

The sum in (1) is over all combinatorially inequivalent 2-complexes, whether or not they come from a triangulation. For simplicity, we begin by assuming that the vertices of J are 5-valent and the edges are 4-valent. (That is, we assume that $N(J)$ vanishes unless these conditions are satisfied). This restriction is not really necessary for what follows, and we shall later indicate how to drop it, but it simplifies presentation substantially.

The “color” $c = \{a_f, b_e\}$ is an assignment of a unitary irreducible representation a_f of a Lie group G to each face f of J , and the assignment of an “intertwiner” b_e to each edge e of J . Intertwiners are defined below. A “spin foam” is a colored 2-complex, namely a couple (J, c) .

Finally, $A_v(c)$ is a given function of the a_f ’s and the b_e ’s that color the ten faces and the five edges adjacent the vertex v . For each fixed 5-valent vertex v , we denote the colors of the five adjacent vertices as b_i^v and the colors of the ten adjacent faces as a_{ij}^v , with $i < j$. Indices i, j, k take value $1, \dots, 5$ throughout the paper. Then

$$A_v(c) = A(a_{ij}^v, b_k^v). \quad (2)$$

Often in the literature, equation (1) is written with a weight associated to the edges as well. However, since each edge is bounded by two vertices, this weight can always be absorbed in A_v .

A specific model is determined by choosing a group G , a vertex function $A(a_{ij}, b_k)$, and the weight of each 2-complex $N(J)$. Various generalizations are possible. We already mentioned that vertices and edges of different valence can be considered. Also, the a ’s and b ’s may be representations and intertwiners of a quantum group.

As mentioned in opening, a surprising number of approaches to quantum gravity, following very different paths, have converged to formulations of this kind. For instance: In loop quantum gravity, the spin foams (J, c) emerge [4,5] as histories of spin networks [21], that is, histories of quantum states of the space geometry [22,23]. The vertex amplitude $A_v(c)$ is given by the matrix elements of the hamiltonian constraint, and a spin foam

has a natural interpretation as a discretized quantized spacetime. In covariant lattice approaches, the sum over colors is the integration over group elements associated to links, expressed in a (“Fourier”) mode expansion over the group. In this case, as we shall see in detail in Section IV, the vertex amplitude $A_v(c)$ is a discretized version of (the exponent of) the lagrangian density [13,16]. In topological field theories, the vertex function is a natural object in the representation theory of the group G , satisfying a set of identities that assure triangulation independence [8–10]. Finally, in the modifications of topological quantum field theories related to quantum general relativity [11–13], the topological field theory vertex amplitude is altered in order to incorporate a quantum version of the constraints that reduce the BF topological field theory [24] to general relativity [25].

A. Intertwiners, the Turaev-Ooguri-Crane-Yetter and the Barrett-Crane models

Given a face f colored with the representation a_f , we associate to f the Hilbert space $H_f = H_{a_f}$, the Hilbert space on which a_f is defined. Given an edge e , bounded by the faces $f_1 \dots f_4$, we associate to e the Hilbert space

$$H_e = H_{f_1} \otimes \dots \otimes H_{f_4}. \quad (3)$$

H_e decomposes in orthogonal subspaces that transform according to different representations of G . Let H_e^0 be the invariant subspace (the trivial representation subspace). Pick, once and for all, a basis $(b^{(1)}, \dots, b^{(n)})$ in H_e^0 . An intertwiner is an element of this basis. (The notation does not explicitly indicate the fact that an intertwiner b depends on the colors a_{f_u} of its four adjacent faces, in the sense that it is an element of a Hilbert space determined by such colors.)

Consider now a vertex v bounded by the five edges $e_1 \dots e_5$, in turn bounded by the ten faces f_{ij} . We associate to v the Hilbert space

$$H_v = H_{e_1} \otimes \dots \otimes H_{e_5}. \quad (4)$$

Notice that $H_v = K \otimes K$, where

$$K = H_{f_{12}} \otimes \dots \otimes H_{f_{45}}. \quad (5)$$

The scalar product on K determines naturally a trace on H_v

$$Tr(v \otimes w) = (v, w) \quad v, w \in K. \quad (6)$$

There is thus a quantity naturally associated to a vertex of v a colored 2-complex, which is

$$A^{TOCY}(a_{ij}, b_i) = Tr(b_1 \otimes \dots \otimes b_5). \quad (7)$$

The simplest spin foam model is the Turaev-Ooguri-Crane-Yetter (TOCY) model [8–10], which is defined by

the following choices. First, the group G is chosen to be the quantum group $SU(2)_q$ or the Lie group $SU(2)$. Second, the vertex function is A^{TOCY} , given in (7). Finally, $N(J)$ vanishes for all J except the single 2-complex $J(\Delta)$ determined by a triangulation Δ of a given 4d manifold M . One can then prove that with these choices Z is independent from the triangulation chosen, and depends on the manifold only. The sum over colorings diverges for $SU(2)$ and is finite for $SU(2)_q$. The model can be obtained as a quantization of 4d BF theory [9,24]. Triangulation independence reflects the fact that BF theory is topological, namely has only a finite number of degrees of freedom. The 3d version of this model is the celebrated Turaev-Viro state sum [26]; its Lie group version is the Ponzano-Regge [27] formulation of 3d quantum gravity. In turn, Ponzano Regge quantum gravity is essentially the covariant version of 3d loop quantum gravity, as it was early recognized in [28].

We can chose an orthonormal basis in each representation space H_f . Vectors in H_f can then be written in terms of their components v^α , $\alpha = 1, \dots, \dim(a_f)$. Vectors in H_e can then be written as $v^{\alpha_1 \dots \alpha_4}$, where the four indices live in the four representations associated to the four faces that e bounds. In particular, an intertwiner b has the form $b^{\alpha_1 \dots \alpha_4}$, and is invariant under the action of the group

$$R^{a_1 \alpha_1}_{\beta_1}(g) \dots R^{a_4 \alpha_4}_{\beta_4}(g) b^{\beta_1 \dots \beta_4} = b^{\alpha_1 \dots \alpha_4} \quad \forall g \in G. \quad (8)$$

where $R^{a\alpha}_\beta(g)$ is the representation matrix in the representation a of G . The normalization and orthogonality conditions on the intertwiners read

$$b^{\alpha_1 \dots \alpha_4} b'^{\alpha_1 \dots \alpha_4} = \delta_{bb'}. \quad (9)$$

(Repeated indices are summed.) A vector in H_v has twenty indices: two for each representation a_{ij} coloring the faces adjacent to the vertex v . The trace (6) simply contracts the pairs of indices in the same representation. In particular

$$A^{TOCY}(a_{ij}, b_k) = \begin{matrix} b_1^{\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}} & b_2^{\alpha_{12}\alpha_{23}\alpha_{24}\alpha_{25}} & b_3^{\alpha_{13}\alpha_{23}\alpha_{34}\alpha_{35}} \\ b_4^{\alpha_{14}\alpha_{24}\alpha_{34}\alpha_{45}} & b_5^{\alpha_{15}\alpha_{25}\alpha_{35}\alpha_{45}} \end{matrix}. \quad (10)$$

If we represent each tensor b as a vertex with four lines –one line per index– and we connect lines of indices summed over (Penrose tensor notation) the right hand side of the above equation yield a 4-simplex, as in Figure 1. More precisely, it yields a graph, which we call Γ_5 that has five 4-valent nodes and ten links. Each link connects two distinct nodes, and the graph is the one-skeleton of a 5-simplex.

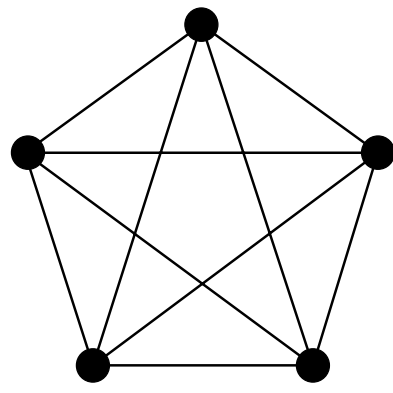


FIG. 1. Structure of the vertex function $A^{TOCY}(a_{ij}, b_k)$: the graph Γ_5 .

Since in the sequel we have various expressions with many indices as in (10), we introduce a more compact notation. We write sequences of indices with running subindices as indices with subindices in parenthesis. That is, for example:

$$b^{\alpha^1_{(i)}} = b^{\alpha^1_2, \alpha^1_3, \alpha^1_4, \alpha^1_5}. \quad (11)$$

Then we can write, for instance

$$A^{TOCY}(a_{ij}, b_k) = \prod_i b^{\alpha^i_{(j)}} \prod_{i < j} \delta_{\alpha^i_j \alpha^j_i}, \quad (12)$$

where, notice, the indices of the two products match.

Finally, we describe the Barrett-Crane (BC) model [11]. The group is $SO(4)$. The representations of $SO(4)$ can be labeled by two half integers j and j' (corresponding to the transformation properties of the representation under the two $SU(2)$ subgroups). The representations of $SO(4)$ for which $j = j'$ are called *simple*. Given two simple representations a_1 and a_2 , the Hilbert space $H_{12} = H_{a_1} \otimes H_{a_2}$ decomposes in a sum over simple as well as non simple representations. Let P_{12} be the projector, defined on H_{12} , over its subspace transforming according to simple representations. Given an edge e , consider the projector

$$P_e = P_{12} P_{13} P_{14} P_{23} P_{24} P_{34}. \quad (13)$$

One can prove [29] that P_e projects over a one-dimensional subspace of H_e . Let b^{BC} be the normalized vector in this one dimensional subspace. Then the Barrett-Crane model is defined by the vertex amplitude

$$A^{BC}(a_{ij}, b_k) = \text{Tr}(b_1^{BC} \otimes \dots \otimes b_5^{BC}). \prod_k \delta_{b_k b_k^{BC}}. \quad (14)$$

There are a number of indications that this model can be related to euclidean quantum general relativity. In fact, the restriction to simple representations can be viewed as an implementation of the constraints that reduce BF theory to general relativity [25].

Unlikely the TOCY model, the BC model is *not* topological. This is appropriate for a model related to quantum general relativity because general relativity is diffeomorphism invariant but has an infinite number of degrees of freedom [18]. The sum depends therefore non trivially from the 2-complex (or the triangulation), and to restore general covariance we have to sum over such structures. A natural extension of the model to a sum over 2-complexes was defined in [20]. The factor $N(J)$ is then given by

$$N(J) = \frac{\lambda^{n(J)}}{\text{Sym}(J)} \quad (15)$$

where $n(J)$ and $\text{Sym}(J)$ are the number of vertices and the number of symmetries of J (see [20]).

III. FIELD THEORY

Given a compact group G and a vertex function $A(a_{ij}, b_k)$, consider the following function over G^{10}

$$W(g_{ij}) = \sum_{a_{ij}, b_k} \bar{\psi}_{a_{ij}, b_k}(g_{ij}) A(a_{ij}, b_k). \quad (16)$$

where g_{ij} is defined for $i < j$ only. Here ψ_{a_{ij}, b_k} is a normalized spin network state on the graph Γ_5 described above, and in Figure 1, with nodes colored by the intertwiners b_i and links colored by the representations a_{ij} . The functions $\psi_{a_{ij}, b_k}(g_{ij})$ form an orthonormal basis in the Hilbert space $L_2[G^{10}/G^5]$ naturally associated to this graph. This is the Hilbert space of a lattice gauge theory on this graph: the Hilbert space of the Haar square integrable functions $\psi(g_{ij})$ of ten group elements g_{ij} , invariant under the five gauge transformations associated to the five vertices of Γ_5

$$g_{ij} \longrightarrow \rho_i g_{ij} (\rho_j)^{-1} \quad (\rho_i \in G). \quad (17)$$

Explicitly, the basis is given by

$$\psi_{a_{ij}, b_k}(g_{ij}) = \prod_{i < j} \dim(a_{ij}) R^{a_{ij} \alpha_{ij} \alpha_{ji}} \prod_k b_k^{\alpha_{k(i)}}. \quad (18)$$

From $W(g_{ij})$, we define a function of twenty group elements h_j^i (with $i \neq j$) by

$$V(h_j^i) = W\left(h_j^i (h_i^j)^{-1}\right). \quad (19)$$

(To visualize this step, consider the graph Γ_5 . Cut its ten links l_{ij} into two parts, which we denote l_j^i and l_i^j . The resulting graph, $\tilde{\Gamma}_5$ has twenty links, five 4-valent nodes and ten 2-valent nodes. Orient each link from the 4-valent node to the 2-valent node, and associate a group element h_j^i to each link l_j^i . See Figure 2.)

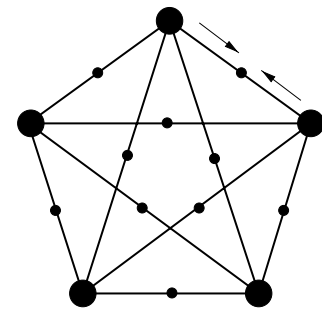


FIG. 2. The structure of the vertex function $V(h_j^i)$: the graph $\tilde{\Gamma}_5$.

Consider then a field theory for a real scalar field $\phi(h_1, \dots, h_4)$ over G^4 , defined by the action

$$S[\phi] = \int_{G^4} dh_u \phi^2(h_{(u)}) + \frac{\lambda}{5!} \int_{G^{20}} dh_j^i V(h_j^i) \prod_k \phi(h_{(i)}^k). \quad (20)$$

Indices u, v take value $1, \dots, 4$. We also assume that $\phi(h_1, \dots, h_4)$ is G invariant,

$$\phi(g_1, g_2, g_3, g_4) = \phi(gg_1, gg_2, gg_3, gg_4), \quad (\forall g \in G); \quad (21)$$

and symmetric

$$\phi(g_1, g_2, g_3, g_4) = \phi(g_{\sigma(1)}, g_{\sigma(2)}, g_{\sigma(3)}, g_{\sigma(4)}) \quad (\forall \sigma), \quad (22)$$

where σ is a permutation of four elements. These assumptions can be dropped by appropriately adjusting the quadratic term in the action; we keep them for simplicity. We have then the following

Main result: The formal Feynman perturbation series of the partition function of this theory

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} \quad (23)$$

is given precisely by (1), with $N(J)$ given by (15).

More in detail, the Feynman graphs of the theory are in 1-1 correspondence with the two complexes J ; the momenta of the field are in 1-1 correspondence with the a 's and b 's (they are discrete because the group is compact), and for each Feynman graph the sum over momenta is precisely the sum over colorings in (1).

To prove this result, we expand the fields in modes over the group, using Peter-Weyl theorem.

$$\phi(g_{(u)}) = \sum_{a_u \alpha_u \beta_u} \phi_{a_{(u)} \alpha_{(u)} \beta_{(u)}} \prod_u R^{a_u \alpha_u \beta_u}(g_u). \quad (24)$$

It is not difficult to see [20] that gauge invariance of the field required by equation (21) implies that the field can be written as

$$\phi(g_{(u)}) = \sum_{a_u b \alpha_u} \phi_{a_{(u)} b \alpha_{(u)}} b^{\beta_{(u)}} \prod_u (\dim a_u) R^{a_u \alpha_u \beta_u}(g_u). \quad (25)$$

Inserting this expansion in the action, we can read out propagator and vertex. The calculation is straightforward, and based only on the orthogonality of the representation matrices

$$\int dg \overline{R^{a\alpha}_{\beta}(g)} R^{a'\alpha'}_{\beta'}(g) = \frac{1}{\dim a} \delta_{aa'} \delta^{\alpha\alpha'} \delta_{\beta\beta'}. \quad (26)$$

The propagator turns out to be (recall the field is symmetric)

$$P^{a_u \alpha_u; a'_u \alpha'_u} = \sum_{\sigma} \prod_u \delta^{a_u \sigma(a'_u)} \delta^{\alpha_u \sigma(\alpha'_u)}. \quad (27)$$

And the vertex

$$V^{a_j^i b_i \alpha_j^i} = A(a_j^i, b_i) \prod_{i < j} \delta^{a_j^i a_i^j} \delta^{\alpha_j^i \alpha_i^j}. \quad (28)$$

The structure of the deltas in the propagator and in the vertex is illustrated in Figure 3.

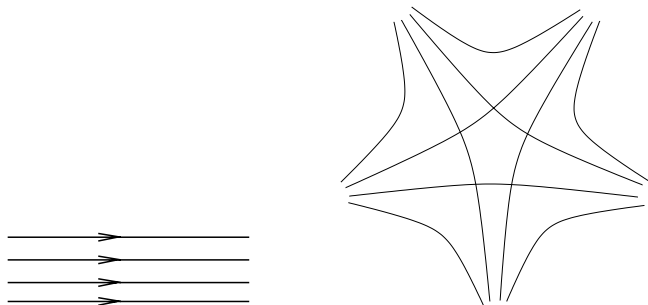


FIG. 3. The structure of the deltas in the propagator and in the vertex.

A Feynman graph is obtained by taking n vertices and contracting them with propagators. Let us call e_1 one of these propagators. Each end of the propagator has four α indices (see (27)). The vertex is five-valent and has twenty α indices (see (28)). When contracting one of the α indices of e_1 with a propagator, this index hits one of the delta's $\delta^{\alpha_j^i \alpha_i^j}$ in (28). That is, it gets contracted with one of the α indices of a second propagator. Call this second propagator e_2 . But the propagator e_2 , in turns, contains a delta function, which connects the index to a second vertex. We can thus follow the contraction along the graph, obtaining a sequence of edges $(e_1, e_2, e_3 \dots)$. Since the graph is finite, the sequence must close to itself. In summing the indices of the last delta we obtain the number $\delta^{\alpha_i^j \alpha_j^i} \delta_{\alpha_i^j \alpha_j^i} = \dim(a_{ij})$. (Since, because of the

delta's $a_j^i = a_i^j$, we can forget the order between i and j and denote this representation as $a_{ij}, i < j$.) We can thus drop all the α indices all together, and add to the sum a factor $\dim(a_{ij})$ for each cycle of edges. The sum over permutations is then converted in a sum over all ways of writing cycles over the graph. But a graph with cycles of edges is precisely a 2-complex. The sum is over graph becomes thus a sum over 2-complexes, in which the representations a_{ij} label the faces, and the intertwiners b_i 's label the edges. The amplitude is obtained by multiplying the vertex amplitudes, and the factor $\dim a_{ij}$ for every face. Finally, the weight of each graph is given by standard Feynman-graphology as the coupling constant to the power of the number of vertices divided by the symmetry factor of the 2-complex. This completes the proof of our main result.

Let us consider an example. The vertex function for the $SU(2)$ TOCY model, $V^{TOCY}(g_{ij})$, is obtained by first inserting $A^{TOCY}(a_{ij}, b_i)$, defined in (7) into the equation (16). The result is that $W^{TOCY}(g_{ij})$ is the distribution with support on the group elements $g_{ij} = 1$ and their gauge equivalents:

$$W^{TOCY}(g_{ij}) = \int_{G^i} dq_i \delta(q_i g_{ij} (q_j)^{-1}), \quad (29)$$

where

$$\int_G dg \delta(g) f(g) = f(1). \quad (30)$$

To prove (29), it is sufficient to integrate its two sides against all basis elements in the Hilbert space, and notice that in both cases we get

$$\int_{G^{10}} dg_{ij} W^{TOCY}(g_{ij}) \psi_{a_{ij}, b_k}(g_{ij}) = A(a_{ij}, b_k). \quad (31)$$

Inserting $W^{TOCY}(g_{ij})$ in (19) and in the action (20), ten of the twenty integrations can be performed immediately, because the field is gauge invariant. This gives the action

$$S[\phi] = \int_{G^4} dh_u \phi^2(h_{(u)}) + \frac{\lambda}{5!} \int_{G^{10}} dg_{ij} \prod_j \phi(g_{ij}), \quad (32)$$

where $g_{ji} = (g_{ij})^{-1}$. This is precisely the Ooguri action [9], or, in three dimensions, the Boulatov action [30], from which the TOCY model was derived in the first place.

IV. GEOMETRICAL INTERPRETATION AND LATTICE THEORY

We now give a geometrical interpretation to the above construction. This interpretation connects the spin foam formulation with lattice gauge theory.

To this purpose, let us analyze the Feynman expansion in the “coordinate” g space, instead than in the “momentum” space of the modes (a, b) . The Feynman expansion

of our field theory is still given as a sum over Feynman graphs Γ , with five valent nodes

$$Z = \sum_{\Gamma} Z(\Gamma), \quad (33)$$

For each graph Γ , we have four group elements at each end of each propagator. Namely, we have four plus four group elements for each edge e of the graph. We denote the group elements at the two ends of the edge e as h_u^e and $h'_u{}^e$, where, we recall $u = 1, \dots, 4$. We have then immediately

$$Z(\Gamma) = \int \prod_e dh_u^e dh'_u{}^e \prod_e P(h_u^e, h'_u{}^e) \prod_v V(h_j^i), \quad (34)$$

where v labels the vertices of the graph, and the twenty group elements h_j^i in the argument of $V(\cdot)$ are the five times four group elements associated to the four edges bounded by the vertex v . The propagator corresponding to the edge e is

$$P(h_u^e, h'_u{}^e) = \sum_{\sigma} \prod_u \delta(h_u^e, h'_{\sigma(u)}{}^e), \quad (35)$$

so that half of the group integrals can be performed immediately, leaving

$$Z(\Gamma) = \int \prod_e dh_u^e \prod_v \sum_{\sigma_e} V(h_{\sigma_e(u)}^e). \quad (36)$$

where σ_e are the permutations of the u 's in h_u^e . There are now only four group elements $dh_u^e, u = 1, \dots, 4$ associated to each edge e .

Now, due to equation (19), the function V depends on ten products only, out of the twenty h_u^e . That is, the twenty group elements in the argument of V get paired. Let us number the edges around a vertex as 1 to 5, and, in each vertex, denote the group elements as $h_j^i, i, j = 1, \dots, 5$. Here the upper index i denotes the edge to which the group element belongs, and the lower index j denotes (four each fixed permutation) the edge to which it is paired. Thus h_j^i enters $V(h_j^i)$ only through the combination $g_{ij} = h_j^i (h_i^j)^{-1}$. For each given set of permutations, group elements get paired across the vertices. Precisely as we did in momentum space, we can thus replace the assignment of a fixed set of permutations with the assignment of all the cycles generated in this manner. We identify cycles as faces. A graph with a full set of cycles is thus a graph with faces, namely a 2-complex. Therefore the sum over graphs and the sum over permutations combine in a sum over 2-complexes J , and we obtain

$$Z = \sum_J Z(J), \quad (37)$$

where the complex amplitude is

$$Z(J) = \int \prod_f dh_f^e \prod_v V(h_f^e). \quad (38)$$

Where, now the group elements are associated to edges e and adjacent faces f .

We can now get to the geometrical interpretation of equation (38). Pick a 2-complex J . There is naturally a lattice L which is, in a sense, “dual” to J . To construct the lattice L , imagine that the 2-complex J is formed by actual surfaces f immersed in a manifold, joining at the (4-valent) edges e (segments in M), which, in turn, join at the (5-valent) vertices v (points in M). In other words, let us consider J not as an abstract combinatorial set, but as 2-dimensional subset of a manifold. Now, pick a point p_f on each surface f and a point p_e on each edge e , and draw an (oriented) link l_f^e that goes from each p_f to each of the p_e in the edges that bound the face f . The collection of all these links forms a lattice (a graph), which we call L . Notice that the nodes of the graph L are of two kinds: the nodes p_e are 4-valent, while the nodes p_f can have arbitrary valence, because a face can be bound by an arbitrary number of edges. Each link is oriented from 4-valent p_e node to an n -valent p_f node.

The lattice L has additional structure, deriving from the vertices of the original 2-complex J . Consider a vertex v of J . The vertex v is in the boundary of five edges e_i and ten faces f_{ij} . Accordingly, there is a portion of the lattice L which is “around” v : the portion formed by the twenty links $l_j^i = l_{f_{ij}}^{e_i}$. We call this portion of the lattice the “elementary” lattice, and denote it as L_v . L_v is a small lattice formed by twenty oriented links. Each link emerges from one of five 4-valent nodes, and the links joins in pairs at 2-valent nodes: l_j^i joins l_i^j . (The 2-valent nodes are the n -valent nodes p_f in L , of which two links only belong to L_v .) This is precisely the $\tilde{\Gamma}_5$ graph of Figure 2. The full lattice L is formed by putting together many elementary lattices L_v . Two elementary lattices L_v are joined by putting in common, and identifying a 4-valent node p_e , and its four links l_u^e . This, is clearly the operation of joining two vertices with an edge, seeing in the dual picture.

Let us now consider a lattice gauge theory on the lattice L . We associate a group element h_f^e to each link l_f^e of the lattice. By locality, the action of the theory $S[h_f^e]$ must be a sum of the discretized lagrangian density $\mathcal{L}_v(h_j^i)$ of each elementary lattice L_v . The partition function is thus

$$Z(L) = \int \prod_f dh_f^e \exp\{\sum_v \mathcal{L}_v(h_j^i)\}. \quad (39)$$

which is precisely (38) with

$$V(h_j^i) = e^{\mathcal{L}_v(h_j^i)}. \quad (40)$$

Therefore the partition function of our field theory can be seen as a lattice gauge theory, defined over the lattice L , and then summed over all possible lattices.

The above construction becomes much more clear in the case in which J is the 2-skeleton of the dual of a 4d triangulation Δ of a 4d manifold M . In this case the elementary lattices L_v are simply the 4-simplices of Δ . More precisely, L_v is a graph on the boundary of the 4-simplex. The boundary of a 4-simplex is a 3d compact space (a 3-sphere), triangulated by five tetrahedra e_i (dual to the edges), separated by ten triangles f_{ij} (dual to the surfaces). The points p_e sit in the center of each tetrahedra. If we join p_{e_1} and p_{e_j} with a segment, the segment must cross a triangle, in a point, which we call $p_{f_{ij}}$. Notice that L_v is precisely the graph on the boundary of the 4-simplex on which the “boundary data” of reference [13] are given. Thus, the potential $V(h_j^i)$ can be seen as the amplitude for all elementary tetrahedron of the triangulation, given as a function of the boundary data. We illustrate the relation between the elementary lattice $\tilde{\Gamma}_5$ and the 4-simplex in Figure 4, by going one dimension down, namely by representing a 3-simplex, that is, a tetrahedron, and the corresponding elementary lattice $\tilde{\Gamma}_4$ on its boundary.

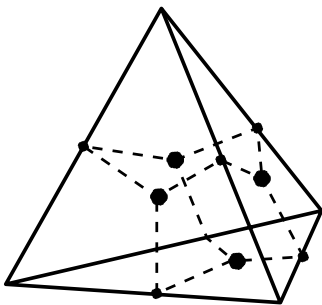


FIG. 4. The n -simplex and the elementary lattice $\tilde{\Gamma}_{n+1}$ on its boundary, here illustrated in the $n = 3$ case.

We close this section by returning to the example provided by the TOCY model. For this model, the potential is the gauge invariant extension of the delta function on the group. Thus, the corresponding lattice gauge theory is obtained by integrating over all group elements such that the holonomy (the product of the group elements) along each loop in each L_v is the unit in the group. In the continuum limit, this is equivalent to integrating over flat connections A . The restriction to flatness can be obtained with a lagrange multiplier B multiplying the curvature F of A , and thus the continuum limit of the theory is of the form

$$Z = \int DA DB e^{\int B \wedge F}. \quad (41)$$

Which illustrates the relation between the TOCY model and BF theory [9].

It is immediate to generalize our construction to spin foam models that have vertices of valence different than five. It suffices to add a potential term of order n in the fields for each kind of n -valent vertex of the spin foam model. It is also immediate to generalize the model to edges of different valence. To obtain this, we have to introduce a different field $\phi(g_1, \dots, g_n)$, with n arguments for each allowed n . In general, the amplitude of a vertex with n edges and m faces will be determined by a potential term with n fields depending, altogether, on m group elements.

In conclusion, we have found that any spin foam model can be obtained from the Feynman expansion of a suitable field theory on a group manifold. In doing that, we have an immediate natural generalization of the spin foam model to a “sum over 2-complexes”. If the model is topological, then the terms of the sum are independent from the 2-complex, and the sum over 2-complexes is a useless complication. However, if the model is not topological, as in models that attempt to construct quantum theory of the gravitational field, the sum over 2-complexes provides a way to recover the covariance broken by the choice of a fixed triangulation and to eliminate the artificial cut off on the number of degrees of freedom introduced by a single triangulation.

In a quantum gravity model, the sum over colored 2-complexes, or spin foams, can be seen as a well-defined version of Hawking’s sum over geometries. Indeed, each colored triangulation can be viewed as a spacetime with its metric. Thus, the field theory representation is the precise 4d analog of the 2d matrix models [31], in which a sum over 2d spacetimes was generated as the Feynman expansion of a suitable matrix theory. In fact, the historical path that has lead to spin foam models from the state sum formulations of topological field theories started precisely from Boulatov’s generalization of the 2d matrix models to 3 dimensions [30].

Canonical quantum general relativity has developed in the loop representation; as mentioned, remarkably the Feynman’s spacetime representation of loop quantum gravity gives precisely a spin foam model. The other way around, the Hilbert space associated to a canonical formulation of (1) can be represented as the kernel of a hamiltonian constraint operator (a Wheeler-DeWitt equation) over a space spanned by a basis of spin networks, precisely as in loop quantum gravity, and, as in loop quantum gravity, the constraint acts locally at the nodes of the spin network [32]. This remarkable convergence opens wide possibilities for exploring the theory with the two complementary tools provided by the covariant and the canonical theory. The representation of the sum over spin foams as a field theory provides a non perturbative handle on the theory, and offers the intriguing possibility of applying standard quantum field theoretical machinery. For instance, conventional renor-

malization theory in the field theory context might be useful for dealing with the potential divergences in the sum. Quantum field theoretical methods might also provide helpful in relation to the problem of extending to the covariant formulation the weave technique [33], namely in identified the coherent quantum states corresponding to a given classical spacetime.

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